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Exponential Stability of Linear Impulsive Differential Equations

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The notion of *exponential stability* for linear impulsive differential equations at fixed moments is made precise.

1. INTRODUCTION

In relation to numerous applications in science and technology, recently the theory of impulsive differential equations has been developed intensively (Lakshmikantham and Liu, 1989; Lakshmikantham *et al.*, 1989; Leela, 1977; Milev and Bainov, to appear; Samoilenko and Perestyuk, 1987; Simeonov and Bainov, 1988). In the present paper the notion of *exponential stability* for linear impulsive differential equations at fixed moments is made precise.

2. PRELIMINARIES

Let $t_0 < t_1 < \cdots < t_i < \cdots$, $\lim t_i = \infty$ as $i \to \infty$, be a given sequence of real numbers. Consider the linear impulsive differential equation (LIDE) at fixed moments

$$\begin{aligned} \frac{dx}{dt} &= A(t)x, \qquad t \neq t_i \\ x(t_i+0) &= B_i x(t_i), \qquad i = 1, 2, \ldots \end{aligned} \tag{1}$$

where the $n \times n$ coefficient matrix $A(t)$ is piecewise continuous in the interval $[t_0, +\infty)$ with points of discontinuity of the first kind at $t = t_i$, $i = 1, 2, \ldots,$

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and the impulse matrices B_i, $i=1, 2, \ldots$, are contant. The underlying vector space is $\mathbb{R}^n$ or $\mathbb{C}^n$.

The solutions $x(t)$ defined in the interval $[t_k+0, +\infty)$ are continuously differentiable for $t \neq t_i$ with points of discontinuity of the first kind at $t=t_i$, $i>k$. Let us note that $x(t_i):=x(t_i-0)$, $i=1, 2, \ldots$. The fundamental matrix $X(t, s)$ of the LIDE (1) for $t \geq s$, $t \in [t_m+0, t_m+1]$, $s \in [t_{j-1}+0, t_j]$, $m \geq j-1$, admits the representation

$$X(t, s) = U(t)U^{-1}(t_m+0)B_m U(t_m) \cdots U^{-1}(t_j+0)B_j U(t_j)U^{-1}(s) \quad (2)$$

where $U(t)$ is the fundamental matrix of the equation $dx/dt = A(t)x$. The fundamental matrix is invertible if and only if the impulse matrices B_i, $j \leq i \leq m$, are nonsingular.

Definition 1. The LIDE (1) is said to be:

(a) *Stable* if for any $\varepsilon > 0$ and for any $s \geq t_0$ there exists $\delta > 0$ such that for each solution x for which $|x(s)| < \delta$ the inequality $|x(t)| < \varepsilon$ holds for $t \geq s$.

(b) *Uniformly stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $s \geq t_0$ and for each solution x for which $|x(s)| < \delta$ the inequality $|x(t)| < \varepsilon$ is valid for $t \geq s$.

Definition 2. The LIDE (1) is said to be:

(a) *Equiasymptotically stable* if it is stable and, moreover, for any $s \geq t_0$ there exists $\eta = \eta(s) > 0$ and for any $\varepsilon > 0$ there exists $T > 0$ such that for each solution x for which $|x(s)| < \eta$ the inequality $|x(t)| < \varepsilon$ holds for $t \geq s + T$.

(b) *Uniformly asymptotically stable* if it is uniformly stable and, moreover, there exists $\eta > 0$ and for any $\varepsilon > 0$ there exists $T > 0$ such that for each solution x and for any $s \geq t_0$ for which $|x(s)| < \eta$ the inequality $|x(t)| < \varepsilon$ is valid for $t \geq s + T$.

Remark 1. All solutions of the LIDE (1) are stable (uniformly stable, equiasymptotically stable, or uniformly asymptotically stable) if and only if its zero solution enjoys the same property.

3. MAIN RESULTS

Denote by L_k, $k=0, 1, 2, \ldots$, the linear space of solutions $x(t)$ of the LIDE (1) defined in the interval $[t_k+0, +\infty)$. Let $e_j := \mathrm{col}(\delta_1^j, \ldots, \delta_n^j)$, where

$$\delta_i^j = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

is Kronecker's symbol and $\mathrm{col}(\cdots)$ stands for a column vector.

The solutions $x_j(t) = X(t, t_k + 0)e_j$, $j = 1, 2, \ldots, n$, are linearly independent as elements of the vector space L_k. We note that their restrictions to the interval $[t_{k+1} + 0, +\infty)$ as elements of the linear space L_{k+1} are linearly dependent if the impulse matrix B_{k+1} is singular. In this case both the merging of solutions at the point $t_{k+1} + 0$ and the noncontinuability to the left of some solutions of L_{k+1} are observed.

Each solution $x(t)$ with initial value $x(t_k + 0) = \text{col}(\lambda_1, \ldots, \lambda_n)$ is a linear combination of the solutions $x_j(t)$, $j = 1, 2, \ldots, n$,

$$x(t) = X(t, t_k + 0)x(t_k + 0) = \lambda_1 x_1(t) + \cdots + \lambda_n x_n(t) \tag{3}$$

i.e., L_k, $k = 0, 1, 2, \ldots$, are n-dimensional linear spaces.

The classical Definitions 1 and 2 are valid for ordinary differential equations as well. For the LIDE (1) the study of exponential stability is appropriate with the aim of taking into account the specific character of this class of ordinary differential equations.

Definition 3. The LIDE (1) is said to be:

(a) *Exponentially stable* if for any nonnegative integer k there exist positive constants α_k and N_k such that for each solution $x \in L_k$ the following inequality holds:

$$|x(t)| \leq N_k e^{-\alpha_k t} |x(t_k + 0)| \qquad \text{for} \quad t \geq t_k + 0 \tag{4}$$

(b) *Uniformly exponentially stable* if there exist positive constants α and N such that for any nonnegative integer k and for each solution $x \in L_k$ the following inequality is valid:

$$|x(t)| \leq N e^{-\alpha(t-s)} |x(s)| \qquad \text{for} \quad t \geq s \geq t_k + 0 \tag{5}$$

(c) *Weakly exponentially stable* (*weakly uniformly exponentially stable*) with respect to the space of solutions L_k if inequality (4) [inequality (5)] is valid only for the solutions $x \in L_k$, where k is a fixed number.

Remark 2. For the LIDE (1), Definition 1 is equivalent to the following definition (Milev and Bainov, to appear, Propositions 1 and 2).

Definition 4. The LIDE (1) is said to be:

(a) *Stable* if for any nonnegative integer k there exists a positive constant N_k such that for each solution $x \in L_k$ the following inequality holds:

$$|x(t)| \leq N_k |x(t_k + 0)| \qquad \text{for} \quad t \geq t_k + 0 \tag{6}$$

(b) *Uniformly stable* if there exists a positive constant N such that for any nonnegative integer k and for each solution $x \in L_k$ the following inequality holds:

$$|x(t)| \leq N|x(s)| \qquad \text{for} \quad t \geq s \geq t_k + 0 \tag{7}$$

Remark 3. A straightforward verification yields that for the LIDE (1) exponential stability implies stability and uniform exponential stability implies uniform stability.

Proposition 1. If the LIDE (1) is exponentially stable, then it is equiasymptotically stable.

Proof. Let $s \in [t_k + 0, t_{k+1}]$. By the inequality of Gronwall-Bellman,

$$|U(t_k + 0) U^{-1}(s)| \leq \exp \int_{t_k}^{s} |A(\theta)| \, d\theta$$

Choose

$$\eta = N_k^{-1} \exp\left[-\alpha_k s - \int_{t_k}^{s} |A(\theta)| \, d\theta\right]$$

Let ε be an arbitrary positive number. Choose

$$T = \begin{cases} -\alpha_k^{-1} \ln \varepsilon > 0 & \text{for} \quad 0 < \varepsilon < 1 \\ 1 & \text{for} \quad \varepsilon \geq 1 \end{cases}$$

Then for each solution $x \in L_k$ and for any $t \geq s + T$ we have

$$\begin{aligned} |x(t)| &\leq N_k \, e^{-\alpha_k t} |x(t_k + 0)| \\ &= N_k \, e^{-\alpha_k (t-s)} \, e^{\alpha_k s} |U(t_k + 0) U^{-1}(s) x(s)| \\ &< N_k \left[\exp\left[\alpha_k s + \int_{t_k}^{s} |A(\theta)| \, d\theta\right]\right] \eta \exp(-\alpha_k T) \leq \varepsilon \end{aligned}$$

Hence the LIDE (1) is equiasymptotically stable. ■

Remark 4. The inverse assertion is not true. We shall construct an example of an LIDE which is equiasymptotically stable but is not exponentially stable.

Example 1. Let $t_k = k$, $k = 1, 2, \ldots$. Consider the LIDE

$$\begin{aligned} &\frac{dx}{dt} = 0, \qquad t \neq k \\ &x(k+0) = \frac{k-1}{k} x(k-0), \qquad k = 1, 2, \ldots \end{aligned} \tag{8}$$

The solution $x(t)$ can be written in the form

$$x(t) = \frac{[s]}{[t]} x(s)$$

Note that $[k+0]=k$ and $[k-0]=k-1$. A straightforward verification shows that the LIDE (8) is equiasymptotically stable but not exponentially stable, since

$$\limsup_{t \to +\infty} \frac{1}{t} \ln|x(t)| = 0$$

Proposition 2. Definition 2(b) is equivalent to Definition 3(b).

Proof. Let the LIDE (1) be uniformly asymptotically stable. For any solution $x \neq 0$ and for any fixed s of the definition domain of x there exists a positive constant c such that $c|x(s)| = \eta/2$. Since cx is a solution, too, then by Definition 2(b) for any $t \geq T$ we have

$$c|x(t+s)| < \varepsilon = c2\varepsilon\eta^{-1}|x(s)|$$

i.e.,

$$|x(t+s)| < 2\varepsilon\eta^{-1}|x(s)|$$

Fix ε so that $2\varepsilon\eta^{-1} < e^{-1}$. Thus, there exists a positive constant T such that for any solution x and for any s of the definition domain of x for $t \geq T$ the following inequality holds:

$$|x(t+s)| \leq e^{-1}|x(s)| \tag{9}$$

Hence there exists a positive constant T such that for any nonnegative integer k and for any solution $x \in L_k$ for $s \geq t_k + 0$ and $t \geq T$ inequality (9) is valid. Let $t \geq T$ and let $t \in [mT, (m+1)T]$, where m is a positive integer. Since $t/m \geq T$, then in view of (9),

$$\left| x\left(\frac{t}{m} + s \right) \right| \leq e^{-1}|x(s)|$$

$$\left| x\left(2\frac{t}{m} + s \right) \right| \leq e^{-1} \left| x\left(\frac{t}{m} + s \right) \right|$$

$$\vdots$$

$$\left| x\left(m\frac{t}{m} + s \right) \right| \leq e^{-1} \left| x\left(\frac{m-1}{m} t + s \right) \right|$$

i.e.,

$$|x(t+s)| \le e^{-m}|x(s)| \le e^{-t/T+1}|x(s)|$$

Set $\alpha = 1/T > 0$ and obtain that for $t \ge T$

$$|x(t+s)| \le e\, e^{-\alpha t}|x(s)|$$

Let $t \in [0, T]$. Since the LIDE (1) is uniformly stable, then by Definition 4(b) there exists a positive constant $\tilde{N}$ such that

$$|x(t+s)| \le \tilde{N}|x(s)| \le \tilde{N}\, e^{\alpha T}\, e^{-\alpha t}|x(s)|$$

Hence there exist positive constants $\alpha = 1/T$ and $N = \max(e, \tilde{N}\, e^{\alpha T})$ such that for any nonnegative integer k and for any solution $x \in L_k$ the following inequality is valid:

$$|x(t+s)| \le N\, e^{-\alpha t}|x(s)| \qquad \text{for} \quad t \ge 0 \quad \text{and} \quad s \ge t_k + 0$$

i.e., the LIDE (1) is uniformly exponentially stable.

The inverse assertion follows from inequality (5). Choose $\eta = 1/N$, $T = -\alpha^{-1} \ln \varepsilon > 0$ for $0 < \varepsilon < 1$ or $T = 1$ for $\varepsilon \ge 1$ and obtain that

$$|x(t)| \le N\, e^{-\alpha(t-s)}|x(s)| < N\, e^{-\alpha T}\eta \le \varepsilon \quad \blacksquare$$

Proposition 3. Let the LIDE (1) be exponentially stable. There exists a positive constant α and for any positive integer k there exist positive constants $\tilde{N}_k$ such that for any solution $x \in L_k$ the following inequality holds:

$$|x(t)| \le \tilde{N}_k\, e^{-\alpha t}|x(t_k+0)| \qquad \text{for} \quad t \ge t_k + 0$$

Proof. Since the LIDE (1) is exponentially stable, then by (4) for any positive integer k and for any solution $x \in L_k$ we have $\chi[x] \le -\alpha_k$, where $\chi[x]$ stands for Lyapunov's characteristic exponent

$$\chi[x] = \limsup_{t \to \infty} \frac{1}{t} \ln|x(t)|$$

Since L_k is a finite-dimensional linear space and for any solution $x \in L_k$ the representation (3) is valid, then

$$\chi[x] \le \max_{1 \le j \le n} \chi[x_j] = -\beta_k \le -\alpha_k < 0$$

Denote by $\tilde{x}_k$, $k = 0, 1, 2, \ldots$, a solution of L_k with the maximal characteristic exponent, i.e., $\chi[\tilde{x}_k] = -\beta_k$. The restriction of the solution $\tilde{x}_k(t)$ to the interval $[t_{k+1}+0, +\infty)$ is an element of the space L_{k+1}, hence $-\beta_k \le -\beta_{k+1}$.

If we suppose that there exist $n+1$ different exponents $-\beta_{k_0} < -\beta_{k_1} < \cdots < -\beta_{k_n} < 0$, then the restrictions of the solutions $\tilde{x}_{k_0}, \tilde{x}_{k_1}, \ldots, \tilde{x}_{k_n}$ to the interval $[t_{k_n}+0, +\infty)$ are elements of the n-dimensional linear space

L_{k_n} and they should be linearly independent since they have different characteristic exponents. Hence, among the exponents β_k, $k=0, 1, 2, \ldots$, there are at most n different and let

$$\max_{k=0,1,2,\ldots} \{-\beta_k\} = -\beta$$

For an arbitrary solution $x \in L_k$ the representation (3) is valid and since $\max_{1\le j\le n} \chi[x_j] \le -\beta$, then for any $\varepsilon \in (0, 1)$ there exists a positive constant N_k such that

$$|x_j(t)| \le N_k^* \, e^{-\beta(1-\varepsilon)t}$$

Hence

$$|x(t)| \le n|x(t_k+0)| N_k^* \, e^{-\beta(1-\varepsilon)t} = \tilde{N}_k \, e^{-\alpha t}|x(t_k+0)|$$

where $\tilde{N}_k = nN_k^*$ and $\alpha = \beta(1-\varepsilon)$. ■

We shall show that there exist LIDE which are exponentially stable but not uniformly asymptotically stable.

Example 2. Let $t_k = e^k$, $k=0, 1, 2, \ldots$, and consider the LIDE

$$\begin{aligned} \frac{dx}{dt} &= (\{\ln t\} - \tfrac{1}{2})x, \qquad t \ne t_k \\ x(t_k+0) &= e^{-t_k} x(t_k), \qquad k = 1, 2, \ldots \end{aligned} \tag{10}$$

where $\{y\} = y - [y]$ is the fractional part of the number y. We note that $\{k+0\} = 0$ and $\{k-0\} = 1$. For $t \ge s$ the solution is written down in the form

$$x(t) = x(s) \exp[t\{\ln t\} - s\{\ln s\} - \tfrac{3}{2}(t-s)]$$

and a straightforward verification yields that the LIDE (10) is exponentially stable but not uniformly asymptotically stable.

Proposition 4. If the LIDE (1) is weakly exponentially stable (weakly uniformly exponentially stable) with respect to the space L_k, then the LIDE (1) is weakly exponentially stable (weakly uniformly exponentially stable) with the same exponent with respect to the spaces L_i, $0 \le i \le k$, as well.

Proof. By the inequality of Gronwall-Bellman for any $\tau_1, \tau_2 \in [t_{t-1}+0, t_k]$ the following inequality holds:

$$|U(\tau_1)U^{-1}(\tau_2)| \le \exp \int_{t_{k-1}}^{t_k} |A(\theta)| \, d\theta = a_k$$

Let the LIDE (1) be weakly exponentially stable with respect to the space L_k. For any solution $x \in L_{k-1}$, its restriction to the interval $[t_k+0, +\infty)$

belongs to the space L_k and by Definition 3(c) for any $t \geq t_k + 0$ we have

$$|x(t)| \leq N_k e^{-\alpha_k t} |x(t_k+0)| = N_k e^{-\alpha_k t} |B_k U(t_k) U^{-1}(t_{k-1}+0) x(t_{k-1}+0)|$$
$$\leq N_k |B_k| a_k e^{-\alpha_k t} |x(t_{k-1}+0)|$$

If $t \in [t_{k-1}+0, t_k]$, then

$$|x(t)| = |U(t) U^{-1}(t_{k-1}+0) x(t_{k-1}+0)|$$
$$\leq a_k |x(t_{k-1}+0)| \leq a_k e^{\alpha_k t_k} e^{-\alpha_k t} |x(t_{k-1}+0)|$$

Choosing $\alpha_{k-1} = \alpha_k$ and $N_{k-1} = \max(N_k |B_k| a_k, a_k e^{\alpha_k t_k})$, we obtain that the LIDE (1) is weakly exponentially stable with respect to the space L_{k-1} as well.

Now let the LIDE (1) be weakly uniformly exponentially stable with respect to the space L_k. For any solution $x \in L_{k-1}$ its restriction to the interval $[t_k+0, +\infty)$ belongs to L_k and for $t \geq s \geq t_k + 0$ inequality (5) is valid.

If $t_{k-1} + 0 \leq s \leq t_k < t$, then

$$|x(t)| \leq N e^{-\alpha(t-t_k)} |x(t_k+0)| = N e^{-\alpha(t-s)} e^{\alpha(t_k-s)} |B_k U(t_k) U^{-1}(s) x(s)|$$
$$\leq N |B_k| a_k e^{\alpha(t_k - t_{k-1})} e^{-\alpha(t-s)} |x(s)|$$

If $t_{k-1} + 0 \leq s \leq t \leq t_k$, then

$$|x(t)| = |U(t) U^{-1}(s) x(s)| \leq a_k e^{\alpha(t_k - t_{k-1})} e^{-\alpha(t-s)} |x(s)|$$

Hence, choosing

$$\tilde{N} = \max(N, a_k e^{\alpha(t_k - t_{k-1})}, N |B_k| a_k e^{\alpha(t_k - t_{k-1})})$$

we obtain that the LIDE (1) is weakly uniformly exponentially stable with respect to the space L_{k-1} as well. ■

Proposition 5. Let the LIDE (1) be weakly exponentially stable (weakly uniformly exponentially stable) with respect to the space L_{k-1} and let the impulse matrix B_k be nonsingular. Then the LIDE (1) is weakly exponentially stable (weakly uniformly exponentially stable) with the same exponent with respect to the space L_k as well.

Proof. Since the impulse matrix B_k is nonsingular, then each solution of L_k is a restriction of a solution of L_{k-1}. Hence, if the LIDE (1) is weakly uniformly exponentially stable with respect to L_{k-1}, then it is weakly uniformly exponentially stable with respect to L_k as well.

Now let the LIDE (1) be weakly exponentially stable with respect to L_{k-1}. Then for $t \geq t_k + 0$ we have

$$
\begin{aligned}
|x(t)| &\leq N_{k-1}\, e^{-\alpha_{k-1} t} |x(t_{k-1}+0)| \\
&= N_{k-1}\, e^{-\alpha_{k-1} t} |U(t_{k-1}) U^{-1}(t_k) B_k^{-1} x(t_k+0)| \\
&\leq N_{k-1} |B_k^{-1}| \left\{ \exp\left[\int_{t_{k-1}}^{t_k} |A(\theta)|\, d\theta \right] \exp(-\alpha_{k-1} t) \right\} |x(t_k+0)| \\
&= N_k\, e^{-\alpha_{k-1} t} |x(t_k+0)|
\end{aligned}
$$

where

$$
N_k = N_{k-1} |B_k^{-1}| \exp\left[\int_{t_{k-1}}^{t_k} |A(\theta)|\, d\theta \right]
$$

Hence the LIDE (1) is weakly exponentially stable with respect to the space L_k as well. ■

Proposition 6. Let the impulse matrices B_i, $i = 0, 1, 2, \ldots$, of the LIDE (1) be nonsingular. If the LIDE (1) is weakly exponentially stable (weakly uniformly exponentially stable) with respect to a fixed space L_k, then the LIDE (1) is exponentially stable (uniformly exponentially stable).

Proof. Proposition 6 is a corollary of Propositions 4 and 5.

Remark 5. If the impulse matrix B_k is singular, then it is possible for the LIDE (1) to be weakly uniformly exponentially stable with respect to the space L_{k-1} but not to be weakly stable with respect to L_k. We illustrate this by the following example.

Example 3. Let $t_i = i$, $i = 0, 1, 2, \ldots$, and consider the LIDE

$$
\begin{aligned}
&\frac{dx}{dt} = Ax, \qquad t \neq t_i \\
&x(t_i+0) = B_i x(t_i), \qquad i = 1, 2, \ldots
\end{aligned}
\tag{11}
$$

where

$$
A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
$$

$$
B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{for} \quad i \geq 2
$$

A straightforward verification yields that the LIDE (11) is weakly uniformly exponentially stable with respect to the space L_0, since the impulse at the moment t_1 crumples the "inconvenient" solutions. The LIDE (11) is not weakly stable with respect to any of the spaces L_k, $k \geq 1$, since on the intervals $[t_k + 0, +\infty)$, $k = 1, 2, \ldots$, the problem coincides with the classical one and the matrix A has an eigenvalue greater than zero.

Remark 6. If in Example 3 we define the impulse matrices by

$$B_i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad i = 10j + 1 \qquad \text{and} \qquad B_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad i \neq 10j + 1$$

$j = 0, 1, 2, \ldots$, then the LIDE (11) becomes uniformly exponentially stable.

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